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Effective lagrangian for 3d $\mathcal{N}=4$ SYM theories for any gauge group and monopole moduli spaces

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Abstract: We construct low energy effective lagrangians for 3d $\mathcal{N}=4$ supersymmetric Yang-Mills theory with any gauge group. They represent supersymmetric σ models at hyper-Kähler manifolds of dimension $4r$ (r is the rang of the group). In the asymptotic region, perturbatively exact explicit expression for the metric are written. We establish the relationship of this metric with the TAUB-NUT metric describing the perturbatively exact effective lagrangians for unitary groups and monopole moduli spaces: the former is obtained out of the latter by a proper hyper-Kähler reduction. We describe in details the reduction procedure for SO / Sp / G_2 gauge groups, where it can also be given a natural interpretation in D-brane language. We conjecture that the exact nonperturbative metrics can be obtained by a similar hyper-Kähler reduction from the corresponding multidimensional Atiyah-Hitchin metrics.

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Contents

1. Introduction

In a very well-known paper [1], Seiberg and Witten have found the exact effective low-energy lagrangian for $4d$ $\mathcal{N}=2$ supersymmetric Yang-Mills theories for the SU(2) gauge group. This result was later generalised to other gauge groups [2, 3]. A related interesting question is what is the form of the effective lagrangian in lower-dimensional "descendants" of $4d$ $\mathcal{N}=2$ SYM theory, the models obtained from it by dimensional reduction.

Going down to 3d, we obtain $\mathcal{N}=4$ (in the 3-dimensional sense) SYM theory. Its effective lagrangian was constructed in the SU(2) case in [4, 5]. It represents a hyper-Kähler supersymmetric σ model on the *Atiyah-Hitchin* manifold [6]. This metric also describes the dynamics of two interacting BPS monopoles [7], the moduli space of classical vacua in SYM theory exactly coinciding with the monopole moduli space. Indeed, the low-energy dynamics in the former involves 3 scalars originating from the components of abelian vector potentials in the original $6d$ $\mathcal{N}=1$ theory in the reduced dimensions and a scalar dual to the 3d photon. And this corresponds to a relative distance of two monopoles and their relative phase.

In the asymptotics when the distance $|\mathbf{r}|$ between the monopoles becomes large, this metric goes over to the TAUB-NUT metric (with negative mass term),

$$2ds^2 = \left(1 - \frac{g^2}{2\pi|\mathbf{r}|}\right) d\mathbf{r}^2 + \frac{\left(d\Psi + \frac{g^2}{2\pi}\omega\right)^2}{\left(1 - \frac{g^2}{2\pi|\mathbf{r}|}\right)}, \tag{1.1}$$

where Ψ is the relative phase of the monopoles and

$$\omega(\mathbf{r}) = \cos\theta \; d\phi \tag{1.2}$$

is the abelian connection describing a Dirac monopole. The factor 2 on the left side is a convention introduced to make contact with eq. (1.3) below.

The Atiyah-Hitchin metric can also be written explicitly it terms of elliptic functions [6]. It differs from (1.1) by a series of exponentially suppressed at large distances terms. This series corresponds to a sum over instantons in the 3d SYM theory.

The asymptotical TAUB-NUT metric (1.1) is singular at $r = g^2/2\pi$. But the full Atiyah-Hitchin metric is not: there is no physical reason for the kinetic energy to become singular when the distance between monopoles becomes small or vanishes. Actually, the AH metric can be *reconstructed* from the requirement that it describes a smooth hyper-Kähler manifold and coincides with eq. (1.1) in the asymptotics.

These results were generalized in ref. [8] to SU(N) groups. Again, the effective lagrangian represents a hyper-Kähler σ model on the generalized Atiyah-Hitchin manifold of dimension $4(N-1)$. This also describes the dynamics of N BPS monopoles [7]. In the asymptotics, one obtains a generalized TAUB-NUT metric

$$ds^2 = A_{ml} d\mathbf{r}_m d\mathbf{r}_l + A^{-1}_{ml} \Lambda_m \Lambda_l \,, \tag{1.3}$$

where A is the following $N \times N$ matrix:

$$\begin{aligned} A_{mm} &= 1 - \frac{g^2}{4\pi} \sum_{l \neq m} \frac{1}{|\mathbf{r}_m - \mathbf{r}_l|} \qquad \text{(no summation over } m) \,, \\ A_{ml} &= \frac{g^2}{4\pi |\mathbf{r}_m - \mathbf{r}_l|} \,, \qquad (m \neq l) \,, \end{aligned} \tag{1.4}$$

and

$$\Lambda_m = d\Psi_m + \frac{g^2}{4\pi} \sum_{l \neq m} \omega(\mathbf{r}_m - \mathbf{r}_l) \,.$$

The explicit expressions for the generalized AH metric were not found yet, but a *conjecture* of existence and uniqueness can be formulated: there is only one smooth hyper-Kähler manifold of dimension $4(N-1)$ with the asymptotics (1.3).

One can also consider the $2d$ and 1d dimensionally reduced versions of the original theory and study the corresponding effective lagrangians there. In the $2d$ case, this was done for SU(2) in ref. [9] and for an arbitrary gauge group in ref. [10]. The effective lagrangian also represents in this case a supersymmetric σ model on a manifold of $4r$ dimensions, where r is the rank of the group. The coordinates of target space correspond to four components of abelian (belonging to the Cartan subalgebra) $6d$ vector potentials associated with the reduced dimensions. In 2 dimensions, there are no dynamical degrees of freedom associated with gauge fields. The model involves four complex supercharges. However, it is not a hyper-Kähler σ model, but rather a *twisted* σ model of the class described in ref. [11]. In the simplest SU(2) case, the bosonic part of the euclidean lagrangian takes the form[1]

$$\begin{aligned} \mathcal{L}^{\rm bos} &= \left[1 - \frac{g_2^2}{2\pi(\bar\sigma\sigma + \bar\phi\phi)}\right] \left[|\partial_\mu\phi|^2 + |\partial_\mu\sigma|^2\right] + \\ &+ \frac{i g_2^2 \epsilon_{\mu\nu}}{2\pi(\bar\sigma\sigma + \bar\phi\phi)} \left[\frac{\sigma}{\bar\phi}(\partial_\mu\bar\sigma)(\partial_\nu\bar\phi) + \frac{\bar\sigma}{\phi}(\partial_\mu\sigma)(\partial_\nu\phi)\right] , \end{aligned} \tag{1.5}$$

[1]This expression differs from [10, eq. (3.23)] by the presence of the factor i, which appears after Wick rotation.

where g_2^2 is the $2d$ gauge coupling constant, σ is expressed into reduced abelian gauge potentials of the original $4d$ theory, and ϕ is the abelian complex scalar field there (if lifting up to $d = 6$, ϕ is also expressed into 5-th and 6-th components of the gauge potentials). The first term in eq. (1.5) describes a certain complex metric (which is *not* Kähler). The second twisted term can be associated with torsion.

The result (1.5) can be obtained by evaluating one-loop diagrams. Higher loops vanish due to supersymmetry. And not only that. In two dimensions, $\mathcal{N} = 4$ supersymmetry together with rotational $O(4)$ invariance of the moduli space rigidly fix the functional form of the metric, which makes the result (1.5) *exact*. Another way to see this is to notice that our $2d$ gauge theory (in contrast to its 3d and $4d$ "parents") does not involve instantons of the usual type and the effective lagrangian does not acquire any nonperturbative contributions.[2]

A similar problem can be posed and solved for the quantum mechanical model obtained after reduction of the original theory down to $(0+1)$ dimensions. This was done in ref. [10]. The effective lagrangian involves $5r$ bosonic variables. It represents a generalization of the nonstandard $\mathcal{N} = 4$ σ model living on 5-dimensional target space, which was constructed in ref. [13].

The only problem that has not been solved yet is constructing the effective lagrangians for 3d theories with nonunitary gauge groups. This is done in the present article. The basic observation which allows one to obtain the results in a simple and universal way is that effective lagrangians in different dimensions are all related to each other. The relationship between $4d$ and 3d effective lagrangians was discussed back in [5]. In refs. [14, 10], a similar relationship between 1d and $2d$ effective lagrangians was exploited to determine the form of the latter. In this paper we start from the $2d$ effective lagrangian and reconstruct the 3d one.

In the next section we illustrate our method by rederiving the results for SU(2) by our method. The generalization to all other groups is rather straightforward. In section 3 we explore in details the groups $\mathrm{Sp}(2r)$, $\mathrm{SO}(N)$ for odd and even N, and G_2 and establish the relationship of the corresponding hyper-Kähler manifolds with monopole moduli spaces. The former are obtained from the latter by the procedure of *hyper-Kähler reduction*. Basically, it consists in our case in imposing certain constraints on monopole dynamical variables which are compatible with equations of motion.

In section 4, we give a natural D-brane interpretation of the results obtained along the lines of [15] with insertion of the proper orientifolds [16].

2. Construction of $\mathcal{L}_{\mathrm{eff}}^{d=3}$

The basic idea is to consider the original SYM theory not on $\mathbb{R}^3$ and not on $\mathbb{R}^2$, but rather on $\mathbb{R}^2 \times S^1$, in the spirit of [5, 9, 14, 10]. Playing with the length L of the circle, one can interpolate between $2d$ and 3d pictures.

[2] One can note, however, that $2d$ non-abelian gauge theories containing only adjoint fields involve the so called Z_N instantons [12]. It would be interesting to pinpoint the precise reason by which they do not contribute in this case.

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The lagrangian (1.5) was obtained after integrating out the charged fields in $2d$ theory. Thinking in $2d$ terms, we have now an infinite number of charged fields representing the coefficients in the Fourier series

$$f(x,y;\ z) = \sum_{k=-\infty}^{\infty} f_k(x,y) e^{2\pi i k z/L} \,. \tag{2.1}$$

The relevant variables in the effective lagrangian are still zero Fourier modes of the neutral fields ϕ, σ and their superpartners. The expression (1.5) is replaced by the infinite sum

$$\begin{aligned} \mathcal{L} = &\left[1 - \frac{g_2^2}{2\pi}\sum_{n=-\infty}^{\infty} \frac{1}{(\bar\sigma_n\sigma_n + \bar\phi\phi)}\right]\left[|\partial_\mu\phi|^2 + |\partial_\mu\sigma|^2\right] + \\ &+ i\sum_{n=-\infty}^{\infty} \frac{g_2^2\epsilon_{\mu\nu}}{2\pi(\bar\sigma_n\sigma_n + \bar\phi\phi)}\left[\frac{\sigma_n}{\bar\phi}(\partial_\mu\bar\sigma)(\partial_\nu\bar\phi) + \frac{\bar\sigma_n}{\phi}(\partial_\mu\sigma)(\partial_\nu\phi)\right], \end{aligned} \tag{2.2}$$

where $\sigma_n = [Z + i(\tau + 2\pi n/L)]/\sqrt{2}$ (and also $\phi = (X+iY)/\sqrt{2}$).[3]

In the limit $L \to 0$, only one term in the sum survives and we are reproducing the previous result. But for large $L \gg g_2^{-1}$ all terms are essential. In the limit $L \to \infty$, we can actually replace the sum by an integral. We obtain

$$\mathcal{L} = \left(\frac{1}{2} - \frac{g^2}{4\pi|\mathbf{r}|}\right)\left[(\partial_\mu\mathbf{r})^2 + (\partial_\mu\tau)^2\right] + \frac{ig^2}{2\pi}\boldsymbol{\omega}(\mathbf{r})\epsilon_{\mu\nu}\partial_\mu\tau\partial_\nu\mathbf{r}\,, \tag{2.3}$$

where $g^2 = g_2^2 L$ is the 3-dimensional gauge coupling constant, $\mathbf{r} = (X,Y,Z)$, and $\omega(\mathbf{r}) = \boldsymbol{\omega}(\mathbf{r})d\mathbf{r}$ is defined in eq. (1.2). The variables $\mathbf{r}$ live on $\mathbb{R}^3$ whereas the variable τ lives on the dual circle, $0 \le \tau \le 2\pi/L$. When L is very large, the size of the dual circle is very small which would normally imply that the excitations related to nonzero Fourier modes of τ would become heavy and decouple. This happens, for example, when the $2d$ effective lagrangian is reconstructed with this method from the 1d one [10]: the latter involves $5r$ dynamic bosonic degrees of freedom, while the former — only $4r$. But in our case it would not be correct just to cross out the terms involving τ. The presence of the twisted term $\propto \epsilon_{\mu\nu}$ prevents us to do it.

To understand it, consider a trivial toy model,

$$\mathcal{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + Bx\dot{y} \quad \Longrightarrow \quad H = \frac{1}{2}\left[p_x^2 + (p_y - Bx)^2\right], \tag{2.4}$$

where $x \in \mathbb{R}^1$, while y is restricted to lie on a small circle, $0 \le y \le \alpha$. The lagrangian (2.4) describes a particle living on a cylinder and moving in a constant magnetic field. Now, if the magnetic field B were absent, the higher Fourier modes of the variable y would be heavy and the low-energy spectrum would be continuous corresponding to free motion along x direction. When $B \neq 0$, for *each* Fourier mode of the variable y, we obtain the *same* oscillatorial spectrum. Only the position of the center of the orbit and not the energy depends on $p_y^{(n)} = 2\pi n/\alpha$.

[3]The fields X, Y, Z should not be confused with the spatial coordinates x, y, z.

Thus, we cannot suppress the variable y in the lagrangian (2.4). Likewise, we cannot suppress the variable τ in eq. (2.3). What we can do, however, is to trade it to another variable using the *duality* trick [11, 17, 18]. Let us write instead of (2.3) another lagrangian

$$\mathcal{L} = \left(\frac{1}{2} - \frac{g^2}{4\pi|\mathbf{r}|}\right)\left[(\partial_\mu \mathbf{r})^2 + B_\mu^2\right] + i\epsilon_{\mu\nu}B_\mu\left[\partial_\nu\Psi + \frac{g^2}{2\pi}\boldsymbol{\omega}(\mathbf{r})\partial_\nu\mathbf{r}\right]. \tag{2.5}$$

Now, integrating e^{-S_E} over $\prod d\Psi$ gives us $\epsilon_{\mu\nu}\partial_\nu B_\mu = 0$ which implies that $B_\mu = \partial_\mu\tau$. Substituting it in eq. (2.5), we reproduce the result (2.3). On the other hand, we can integrate over $\prod dB_\mu$ in eq. (2.5). Doing this, we obtain the σ model lagrangian on the manifold (1.1)![4]

This derivation can be readily generalized for other groups. The effective $2d$ lagrangian for an arbitrary group depends on the variables $\mathbf{r}^a$, τ^a, $a = 1, \ldots, r$. Putting the theory on $\mathbb{R}^2 \times S^1$ and doing the sum over all Fourier harmonics of charged fields (which for large L can be replaced by the integral over $\prod_a d\tau^a$, we obtain the bosonic Euclidean effective lagrangian in the following form

$$\mathcal{L} = \sum_j \left(\frac{1}{c_V} - \frac{g^2}{4\pi|\mathbf{r}^{(j)}|}\right)\left[(\partial_\mu \mathbf{r}^{(j)})^2 + (\partial_\mu\tau^{(j)})^2\right] + \frac{ig^2}{2\pi}\sum_j \boldsymbol{\omega}(\mathbf{r}^{(j)})\epsilon_{\mu\nu}\partial_\mu\tau^{(j)}\partial_\nu\mathbf{r}^{(j)}, \tag{2.6}$$

where $\mathbf{r}^{(j)} = \alpha_j(\mathbf{r}^a)$, $\tau^{(j)} = \alpha_j(\tau^a)$, and the sum runs over all positive roots α_j of the corresponding Lie algebra. Performing the duality transformation and trading $\tau^a \to \Psi^a$, we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\mathbf{r}^a)(\partial_\mu\mathbf{r}^b)Q_{ab} + \frac{1}{2}J_\mu^a Q_{ab}^{-1}J_\mu^b, \tag{2.7}$$

where

$$\begin{aligned} Q^{ab} &= \delta^{ab} - \frac{g^2}{2\pi}\sum_j \frac{\alpha_j^a\alpha_j^b}{|\mathbf{r}^{(j)}|}, \\ J_\mu^a &= \partial_\mu\Psi^a + \frac{g^2}{2\pi}\sum_j \boldsymbol{\omega}(\mathbf{r}^{(j)})\partial_\mu\mathbf{r}^{(j)}\alpha_j^a. \end{aligned} \tag{2.8}$$

The relation $\sum_j \alpha_j^a\alpha_j^b = (c_V/2)\delta^{ab}$ (with the normalisation $\sum_a \alpha_j^a\alpha_j^a = 1$ for the long roots) was used.

3. $\mathcal{L}_{\text{eff}}$ and monopole dynamics

Specialization of eq. (2.7) to particular groups is naturally interpreted in terms of metrics on BPS monopoles moduli spaces.

(i) SU(N). This case was first considered in [8]. There are $N(N-1)/2$ positive roots, $\alpha_{ml}(\mathbf{r}) = \mathbf{r}_m - \mathbf{r}_l, m < l = 1, \ldots, N$, $\sum_m \mathbf{r}_m = 0$. Substituting it in eq. (2.7), we reproduce the result (1.3).[5]

[4] A general statement of ref. [17] is that, if the manifold corresponding to a twisted σ model involves an isometry (the metric and torsion do not depend on some variables), the duality transfromations with respect to these variables brings us onto a hyper-Kähler manifold. See appendix for some further clarifications.

[5] Note that for simply laced SU(N), there is no difference between roots and coroots. There is such difference — and this will be important below — in Sp($2r$) and SO($2r+1$) cases.

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The monopole dynamics is described by the following classical equations of motion [7][6]

$$
\begin{aligned}
\ddot{\mathbf{r}}_l - \frac{g^2}{4\pi}\sum_{m\neq l}^{N}\frac{\ddot{\mathbf{r}}_{lm}}{r_{ml}} + \frac{g^2}{8\pi}\sum_{l\neq m=1}^{N}&\frac{2\left[\dot{\mathbf{r}}_{ml}\times\mathbf{r}_{ml}\right]\cdot\dot{\mathbf{r}}_{ml} - \mathbf{r}_{ml}(\dot{\mathbf{r}}^2_{ml})}{r^3_{ml}} - \\
&- \frac{g}{4\pi}\sum_{m\neq l}(q_{ml})\dot{\mathbf{r}}_{ml}\times\frac{\mathbf{r}_{ml}}{r^3_{ml}} + \frac{1}{8\pi}\sum_{m\neq l}\frac{q^2_{ml}\mathbf{r}_{ml}}{r^3_{ml}} = 0\,, \\
&q_l = gA^{-1}_{lm}\left[\dot{\Psi}_m + \frac{g^2}{4\pi}\sum_{n\neq m}\boldsymbol{\omega}(\mathbf{r}_{nm})\dot{\mathbf{r}}_{nm}\right] = \text{const}\,,
\end{aligned}
\tag{3.1}
$$

where $\mathbf{r}_{ml} = \mathbf{r}_m - \mathbf{r}_l$, $q_{ml} = q_m - q_l$. The equations of motion for the effective lagrangian (2.7) have a similar form, with time derivative being replaced by ∂_μ.

(ii) $\mathrm{Sp}(2r)$. There are r long positive roots $\alpha_m(\mathbf{r}) = \mathbf{r}_m$ and $r(r-1)$ short positive roots $\alpha_{ml}(\mathbf{r}) = (\mathbf{r}_m \pm \mathbf{r}_l)/2$ $(m < l = 1, \dots, r$; $\mathbf{r}_m$ are mutually orthogonal and linearly independent). The metric reads

$$
\begin{aligned}
ds^2 &= \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{4\pi}\sum_{\pm}\sum_{m<l}\frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} - \frac{g^2}{2\pi}\sum_m\frac{(d\mathbf{r}_m)^2}{r_m} + \text{phase part} \\
&\equiv Q_{ml}\,d\mathbf{r}_m d\mathbf{r}_l + \text{phase part}\,.
\end{aligned}
\tag{3.2}
$$

The full metric is restored from eqs. (2.7) and (2.8).

An important observation is that the corresponding effective lagrangian is obtained from the effective lagrangian describing the dynamics of $2r+1$ BPS monopoles numbered by the integers $j = -r, \dots, r$ by imposing the constraints

$$
\begin{aligned}
\mathbf{r}_{-r} + \mathbf{r}_r &= \dots = \mathbf{r}_{-1} + \mathbf{r}_1 = 2\mathbf{r}_0 = 0\,, \\
\Psi_{-r} + \Psi_r &= \dots = \Psi_{-1} + \Psi_1 = 2\Psi_0 = 0\,.
\end{aligned}
\tag{3.3}
$$

We *are* allowed to impose these constraints because they are compatible with the equations of motion (3.1) and also with the equations of motion of the corresponding 3d field theory.

The corresponding metric is hyper-Kähler. It follows from: (i) $\mathcal{N} = 4$ supersymmetry of the original theory, which implies $\mathcal{N} = 4$ supersymmetry of the effective lagrangian (2.7), (ii) the absence of the twisted term there, and (iii) the theorem due to Alvarez-Gaumé and Freedman [19]. One can also demostrate the hyper-Kähler nature of the metric more directly by reproducing the result (3.6) in the framework of the hyper-Kähler reduction procedure described in [20]. The reduction of the Gibbons-Manton metric (1.3) is performed with respect to the symmetry

$$
\Psi_j \to \Psi_j + \alpha_j\,, \quad \Psi_{-j} \to \Psi_{-j} + \alpha_j\,, \quad j = 1, \dots, r\,, \qquad \Psi_0 \to \Psi_0 + \alpha_0\,,
\tag{3.4}
$$

where Ψ_j is the phase variable of the jth monopole. The corresponding moment maps are

$$
\mathbf{r}_0, \mathbf{r}_j + \mathbf{r}_{-j}\,, \qquad j = 1, \dots, r\,,
\tag{3.5}
$$

and the hyper-Kähler reduction is made at zero value of the moment maps.

[6]Here g is interpreted as the monopole magnetic charge. In the quantum problem, q_l are quantized to (integer)$/g$ and are interpreted as the electric charges of the corresponding dyons.

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(iii) SO$(2r+1)$. The system of roots is the same as for Sp$(2r)$ only the long and short roots are interchanged: there are now $r(r-1)$ long roots $(\mathbf{r}_m \pm \mathbf{r}_l)/\sqrt{2}$ and r short roots $\mathbf{r}_m/\sqrt{2}$. The metric reads

$$ds^2 = \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{2\pi\sqrt{2}} \left[\sum_\pm \sum_{m<l} \frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} + \sum_m \frac{(d\mathbf{r}_m)^2}{r_m} \right] + \text{phase part}\,. \tag{3.6}$$

This metric is obtained from the Gibbons-Manton type metric for $2r$ BPS monopoles numbered by the integers $j = -r, \ldots, r,\ j \neq 0$ by imposing the constraints

$$\begin{aligned} \mathbf{r}_{-r} + \mathbf{r}_r &= \cdots = \mathbf{r}_{-1} + \mathbf{r}_1 = 0\,, \\ \Psi_{-r} + \Psi_r &= \cdots = \Psi_{-1} + \Psi_1 = 0 \end{aligned} \tag{3.7}$$

and rescaling ds^2 and g^2. The constraints (3.7) are compatible with the equations of motion. The result (3.6) is also obtained by hyper-Kähler reduction with respect to the symmetry

$$\Psi_j \to \Psi_j + \alpha_j\,, \quad \Psi_{-j} \to \Psi_{-j} + \alpha_j\,, \qquad j = 1, \ldots, r\,. \tag{3.8}$$

The corresponding moment maps are

$$\mathbf{r}_j + \mathbf{r}_{-j}\,, \qquad j = 1, \ldots, r\,, \tag{3.9}$$

and the hyper-Kähler reduction is made at zero value of the moment maps.

Note that we obtained the effective lagrangian for Sp$(2r)$ out of that for SU$(2r+1)$ and not out of SU$(2r)$, as one could naively expect in view of the embedding Sp$(2r) \subset$ SU$(2r)$. Likewise, the moduli space for SO$(2r+1)$ is obtained out of SU$(2r)$ and not SU$(2r+1)$. This is due to the fact that magnetic charges are coupled to coroots rather than roots.

(iv) SO$(2r)$. This Lie algebra is simply laced. The positive roots are $\alpha^\pm_{ml}(\mathbf{r}) = (\mathbf{r}_m \pm \mathbf{r}_l)/\sqrt{2}$ $(m < l = 1, \ldots, r)$

The metric reads

$$ds^2 = \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{2\pi\sqrt{2}} \sum_\pm \sum_{m<l} \frac{(d\mathbf{r}_l \pm d\mathbf{r}_m)^2}{|\mathbf{r}_l \pm \mathbf{r}_m|} + \cdots\,. \tag{3.10}$$

To relate this metric to the Gibbons-Manton one, we need first to introduce a massive deformation of the latter,

$$ds^2 = \sum_m (d\mathbf{r}_m)^2 - \frac{g^2}{4\pi} \sum_{l<m} \frac{(d\mathbf{r}_m - d\mathbf{r}_l)^2}{\sqrt{(\mathbf{r}_m - \mathbf{r}_l)^2 + \lambda_{lm}^2}} + \cdots\,, \tag{3.11}$$

where there are $2r$ monopoles numbered by the integers $j = -r, \ldots, r,\ j \neq 0$. This metric is hyper-Kähler [20]. Assuming that only $\lambda_{m,-m}$ are not zero, sending these parameters to infinity, and performing the hyper-Kähler reduction with respect to the symmetry

$$\Psi_j \to \Psi_j + \alpha_j\,, \quad \Psi_{-j} \to \Psi_{-j} + \alpha_j\,, \qquad j = 1, \ldots, r\,, \tag{3.12}$$

with moment maps

$$\mathbf{r}_j + \mathbf{r}_{-j}\,, \qquad j = 1, \ldots, r\,, \tag{3.13}$$

and at zero value of the moment maps, eq. (3.11) is reduced to eq. (3.10) after a proper rescaling.

(v) G_2 There are three long ($\mathbf{r}_1-\mathbf{r}_2, \mathbf{r}_1-\mathbf{r}_3, \mathbf{r}_2-\mathbf{r}_3$) and three short ($\mathbf{r}_{1,2,3}$) positive roots (the constraint $\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3=0$ being imposed). The metric reads

$$ds^2 = \sum_{m=1}^{3} d\mathbf{r}_m^2 - \frac{g^2}{2\pi}\left(\sum_{m>l=1}^{3} \frac{(d\mathbf{r}_m - d\mathbf{r}_l)^2}{|\mathbf{r}_m-\mathbf{r}_l|} + 3\sum_{m=1}^{3}\frac{d\mathbf{r}_m^2}{|\mathbf{r}_m|}\right) + \cdots . \tag{3.14}$$

It can be obtained out of the metric for Sp(6)

$$ds^2 = \sum_{m=1}^{3} d\mathbf{r}_m^2 - \frac{g^2}{4\pi}\left(\sum_{\pm}\sum_{m>l=1}^{3} \frac{(d\mathbf{r}_m \pm d\mathbf{r}_l)^2}{|\mathbf{r}_m\pm\mathbf{r}_l|} + 2\sum_{m=1}^{3}\frac{d\mathbf{r}_m^2}{|\mathbf{r}_m|}\right) + \cdots \tag{3.15}$$

by rescaling and imposing the [compatible with Sp(6) equations of motion] constraints

$$\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3 = 0\,, \qquad \Psi_1+\Psi_2+\Psi_3 = 0\,. \tag{3.16}$$

Again, we obtained $\mathcal{L}_{\rm eff}$ for G_2 out of $\mathcal{L}_{\rm eff}$ for Sp(6), though G_2 is embedded not into Sp(6), but into the dual algebra SO(7).

The metric eq. (3.14) is obtained from the Sp(6) metric by the hyper-Kähler reduction with respect to symmetry

$$\Psi_1 \to \Psi_1 + \alpha\,, \qquad \Psi_2 \to \Psi_2+\alpha\,, \qquad \Psi_3\to\Psi_3+\alpha \tag{3.17}$$

with moment map

$$\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3 \tag{3.18}$$

at zero value of the moment map.

The effective lagrangian for F_4 can be related to the moduli space of 26 monopoles. (26 is the lowest dimension of a unitary group where F_4 can be embedded. This follows from the fact that the representation **26** of F_4 has the lowest dimension.) E_6 can be embedded into SU(27) and hence the corresponding effective lagrangian is related to the moduli space of 27 monopoles. Now, the shortest representation in E_7 has the dimension 56 and we need at least 56 monopoles in this case. Finally, $E_8 \subset$ SU(248) and we need 248 monopoles. The moduli space of 248 monopoles can also be used as a universal starting point to describe the dynamics of F_4, E_6 and E_7, if following the chain of embeddings $F_4 \subset E_6 \subset E_7 \subset E_8 \subset$ SU(248).

The explicit formulae we have written refer to the asymptotic region where nonperturbative effects are suppressed. The corresponding metrics involve singularities at small $|\mathbf{r}^{(j)}|$. Like for the SU(N) case, a very reasonable conjecture is that these singularities can be sewn up and, for any simple Lie group, there is one and only one smooth hyper-Kähler metric with the asymtotics

$$ds^2 = d\mathbf{r}^a Q_{ab} d\mathbf{r}^b + \cdots . \tag{3.19}$$

It is natural to conjecture that this metric is obtained from the multimonopole Atiyah-Hitchin metrics by the hyper-Kähler reduction with respect to the same symmetries as above, eqs. (3.4), (3.8), (3.12) and (3.17).

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4. Orientifolds

We now discuss the brane pictures behind the results obtained above.

Let us first remind [15] that (the Coulomb branch of) the $\mathcal{N} = 4$ 3d SUSY Yang-Mills with gauge group U(N) [7] is convenient to realize as N parallel $D3$ branes stretched between two parallel $NS5$ branes. One of the directions on $D3$ brane is of microscopic size (the distance between $NS5$ branes) so from the $D3$ branes perspective there is 3d gauge theory. Low energy degrees of freedom are positions of the $D3$ branes on the $NS5$ branes which gives $3N$ scalars (the branes assumed to be solid in this counting) and N photons living on the branes, which gives other N scalars. Forgetting about the common U(1) leaves $3(N-1)$ scalar degrees of freedom. The charged particles corresponding to the roots of U(N) appears as F string states corresponding to the strings stretched between two of the N $D3$ branes, in particular, simple roots correspond to the strings stretched between two adjacent branes of the N $D3$ branes, *elementary* strings. Attributing to the jth brane the element e_j of the orthonormalized Euclidean basis $\{e_j,\ j = 1,\ldots,n\}$ one sees that the elementary F strings inherit vectors of the type of $e_{j+1} - e_j$. It is easy to see that intersections of these vectors are described by SU(N) Dynkin diagrams. One can also check that there is the appropriate amount of supersymmetry in the brane system described.

As explained in [15], with use of the results of [21], the SU(2) monopole moduli space appears if one changes the perspective from $D3$ branes to $NS5$ branes. Two parallel $NS5$ branes give U(2) gauge theory, positions of the N $D3$ branes on the $NS5$ branes give positions of N monopoles and the N scalars dual to photons living on the $D3$ branes give N phases to the monopoles. Forgetting about the center-of-mass coordinate and about the common phase gives $3(N-1)$ dimensional relative moduli space of SU(2) monopoles.

Consider now the case of Sp(r) and of SO($2r+1$).

In the Sp(r) case the brane picture consists of $2r$ $D3$ branes and O^+ orientifold in the middle [16]. Let us number the branes by integers $j = -r,\ldots,r,\ j \neq 0$. The locations of $D3$ branes are symmetric with respect to orientifold, $\mathbf{r}_j = -\mathbf{r}_{-j}$. Again, one attributes to the jth brane ($j > 0$) the element e_j of the orthonormalized Euclidean basis $\{e_j,\ j = 1,\ldots,r\}$ and one defines $e_{-j} = -e_j$. The elementary F strings are of two types - those crossing the orientifold and those not crossing the orientifold. The ones not crossing the orientifold appear only in symmetric pairs under the reflection with respect to the orientifold, while the ones crossing orientifold are required to be self-dual under the reflection. Thus, the elementary strings inherit vectors $2e_1, e_2 - e_1, \ldots, e_j - e_{j-1}$, whose intersections are described by Sp(r) Dynkin diagrams.

In the SO($2r+1$) case the brane picture consists of $2r$ $D3$ branes and $\tilde{O}^-$ orientifold. Everything is quite similar to the case of Sp(r) but the rules for elementary F strings are different. They are of two types - those ending on the orientifold and those not ending on the orientifold. Crossing of the orientifold is not now allowed. All the strings appear only in symmetric pairs under the reflection with respect to the orientifold. The elementary strings this time inherit vectors $e_1, e_2 - e_1, \ldots, e_j - e_{j-1}$, whose intersections are described by SO($2r+1$) Dynkin diagrams.

[7] The common U(1) is standard in the brane pictures and is essentially irrelevant.

Magnetically charged states appear as D strings stretched between the $D3$ branes. The orientifolds introduce different rules for F strings and D strings. Actually, the rules for D strings in the presence of O^+ orientifold are the same as for F strings in the presence of $\tilde{O}^-$ and vice versa. This fits nicely with the metrics eqs.(3.6) and (3.2) in the cases of $\mathrm{Sp}(r)$ and $\mathrm{SO}(2r+1)$.

In the $\mathrm{SO}(2r)$ case the brane picture consists of $2r$ $D3$ branes and O^- orientifold. The elementary F (and D) strings can now cross the orientifold but cannot be self-dual under the reflection with respect to the orientifold. They thus inherit the vectors $e_2+e_1, e_2-e_1, \ldots, e_j-e_{j-1}$ whose intersections are described by $\mathrm{SO}(2r)$ Dynkin diagrams. This is the limit $\lambda \to \infty$ which removes from metric eq. (3.10) the terms which would be present if the self-dual strings were present.

In the case of G_2 the orientifold picture is not known. Considerations of the previous section (cf. eqs. (3.17) and (3.18)) suggest to conjecture that a novel kind of orientifold relating positions of three branes (not of two ones like in SO / Sp case) is necessary.

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A. Twisted, hyper-Kähler, and duality

To make the paper more self-contained, we present here some relevant formulae referring to manifestly supersymmetric description of the twisted σ models and the duality transformation relating them to hyper-Kähler models.

The lagrangian of a twisted supersymmetric σ model generically reads

$$L = \int d^2\theta d^2\bar{\theta}\; \mathcal{K}_{tw}(\Phi^a, \bar{\Phi}^a;\; \Sigma^a, \bar{\Sigma}^a)\,, \tag{A.1}$$

where Φ^a is a $\mathcal{N}=2$ chiral superfield,

$$\bar{D}_+\Phi^a = \bar{D}_-\Phi^a = 0\,, \tag{A.2}$$

and Σ^a is a $\mathcal{N}=2$ twisted chiral superfield,

$$D_+\Sigma^a = \bar{D}_-\Sigma^a = 0\,. \tag{A.3}$$

The lagrangian (A.1) is manifestly $\mathcal{N}=2$ supersymmetric. The requirement of $\mathcal{N}=4$ supersymmetry imposes the constraints

$$\frac{\partial^2}{\partial\Sigma^a\partial\bar{\Sigma}^b}\mathcal{K}_{tw} - \frac{\partial^2}{\partial\Sigma^b\partial\bar{\Sigma}^a}\mathcal{K}_{tw} = 0 \tag{A.4}$$

$$\frac{\partial^2}{\partial\Phi^a\partial\bar{\Phi}^b}\mathcal{K}_{tw} + \frac{\partial^2}{\partial\Sigma^b\partial\bar{\Sigma}^a}\mathcal{K}_{tw} = 0\,. \tag{A.5}$$

The target space metric is

$$ds^2 = \frac{\partial^2 \mathcal{K}_{tw}}{\partial \Phi^a \partial \bar{\Phi}^b} d\Phi^a d\bar{\Phi}^b + \frac{\partial^2 \mathcal{K}_{tw}}{\partial \Sigma^a \partial \bar{\Sigma}^b} d\Sigma^a d\bar{\Sigma}^b \,. \tag{A.6}$$

We emphasize that in spite of $\mathcal{N}=4$ supersymmetry, the target space is not hyper-Kähler and not even Kähler. The lagrangian involves also a torsion term, which can be expressed via $\mathcal{K}_{tw}$ as well.

The twisted potential for the effective lagrangian of $2d$ SYM theory is

$$\mathcal{K}_{tw} = \sum_j \left\{ \frac{1}{2c_V} \left[\bar{\Sigma}^{(j)} \Sigma^{(j)} - \bar{\Phi}^{(j)} \Phi^{(j)} \right] - \frac{g^2}{8\pi} \left[F\left(\frac{\bar{\Sigma}^{(j)} \Sigma^{(j)}}{\bar{\Phi}^{(j)} \Phi^{(j)}} \right) - \ln \Phi^{(j)} \ln \bar{\Phi}^{(j)} \right] \right\}, \tag{A.7}$$

where $F(\eta)$ is the Spence function,

$$F(\eta) = \int_1^\eta \frac{\ln(1+\xi)}{\xi} \, d\xi \,. \tag{A.8}$$

Putting the SYM theory on $\mathbb{R}^2 \times S^1$ and "unwinding" S^1, we obtain another twisted σ model, where $\mathcal{K}_{tw}$ depends only on the sums $Z^a = \Sigma^a + \bar{\Sigma}^a$ and does not depend on the differences. It satisfies the generalized harmonicity condition (A.5).(The constraint (A.4) is satisfied automatically.)

In this case, eq. (A.1) can be rewritten as

$$L = \int d^2\theta d^2\bar{\theta} \left[\mathcal{K}_{tw}(\Phi^a, \bar{\Phi}^a, Z^a) - \sum_a (A^a + \bar{A}^a) Z^a \right], \tag{A.9}$$

where Z^a are real superfields and A^a are conventional chiral superfields. Integrating over $\prod_a dA^a d\bar{A}^a$, we obtain $D^2 Z^a = \bar{D}^2 Z^a = 0$, and that implies that Z^a can be represented as $\Sigma^a + \bar{\Sigma}^a$, where Σ^a are twisted superfields. We are thus reproducing eq. (A.1). On the other hand, integrating out the superfields Z^a, we obtain

$$\int d^2\theta d^2\bar{\theta} \mathcal{K}_{hk}(\Phi^a, \bar{\Phi}^a, X^a) \,,$$

where $X^a = A^a + \bar{A}^a$ and $\mathcal{K}_{hk}$ is related to $\mathcal{K}_{tw}$ by a Legendre transformation

$$\mathcal{K}_{hk}(\Phi^a, \bar{\Phi}^a, X^a) = \mathcal{K}_{tw}(\Phi^a, \bar{\Phi}^a, Z^a) - \sum_a Z^a X^a \,, \tag{A.10}$$

where Z^a as a function of X^a is defined from the equation

$$\frac{\partial \mathcal{K}_{tw}}{\partial Z^a} - X^a = 0 \,. \tag{A.11}$$

Now, $\mathcal{K}_{tw}$ satisfies *linear* generalized harmonicity conditions (A.5). This dictates certain *nonlinear* conditions on the hyper-Kähler potential $\mathcal{K}_{hk}$. For 4-dimensional manifolds, a harmonic function $\mathcal{K}_{tw}$ determines the function $\mathcal{K}_{hk}$ satisfying the so called *Monge-Ampere* equation

$$\det \left\| \begin{matrix} \partial^2 \mathcal{K}_{hk} / \partial \bar{\Phi} \partial \Phi & \partial^2 \mathcal{K}_{hk} / \partial \bar{\Phi} \partial X \\ \partial^2 \mathcal{K}_{hk} / \partial X \partial \Phi & \partial^2 \mathcal{K}_{hk} / \partial X^2 \end{matrix} \right\| = \text{const} \,. \tag{A.12}$$

The nonlinear equation (A.12) is much more diffucult to solve than the Laplace equation. If you will, solving first the Laplace equation for $\mathcal{K}_{tw}$ and applying the Legendre transformation afterwards represents a regular method of finding solutions to the Monge-Ampere equation and, correspondingly, a wide class of hyper-Kähler manifolds [22] (the functions $\mathcal{K}_{hk}$ not depending on $\bar{A} - A$ are thus found). Note that in all nontrivial cases, the solution of the differential equation (A.11) is expressed into certain trancendental functions, for which Ryzhik and Gradstein failed to reserve a symbol.

The hyper-Kähler metric is computed via the standard formulas

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \mathcal{K}_{hk} \,. \tag{A.13}$$

In contrast to $\mathcal{K}_{hk}$, it is often expressed into conventional functions (see e.g. eqs. (1.1) and (1.3)).

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